SOME GRAPHICAL ASPECTS OF FROBENIUS STRUCTURES

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ABSTRACT. We survey some aspects of Frobenius algebras, Frobenius structures and their relation to finite Hopf algebras using graphical calculus. We focus on the 'yanking' moves coming from a closed structure in a rigid monoidal category, the topological move, and the 'yanking' coming from the Frobenius bilinear form and its inverse, used e.g. in quantum teleportation. We discus how to interpret the associated information flow. Some care is taken to cover non-symmetric Frobenius algebras and the Nakayama automorphism. We review graphically the Larson-Sweedler-Pareigis theorem showing how integrals of finite Hopf algebras allow to construct Frobenius structures. A few pointers to further literature are given, with a subjective tendency to graphically minded work.

1. Introduction

- 1.1. **Scope of this chapter.** Frobenius algebras surface at many places in mathematics and physics. Quite recently, using a convenient graphical notation, Frobenius algebras have been used to investigate foundational issues of quantum theory references will be given below. Also, as shown elsewhere in this book, Frobenius algebras emerge in the semantic analysis of natural languages. The aim of this chapter is to present the basic results about Frobenius algebras, their relation to finite dimensional Hopf algebras with special emphasis on using graphical notation. Frobenius structures, which are related to ring theory, need to be considered too. Frobenius structures encode such notions as semi simplicity and separability of rings. We need occasionally to extend the graphical calculus to encode properties of underlying rings, but will not venture properly into 2-categorical notions. Moreover, no new results may be found in this chapter, and far from everything that is known about the subject is covered here. However, in passing we will give some pointers to the literature, which unfortunately is by far to large to be considered completely. Note that references do not indicate an attribution, but unless otherwise stated we merely give the source we use.
- 1.2. **Frobenius' problem.** In the late nineteenth century Ferdinand August Frobenius (1849–1917) and his student Issai Schur– studied the representations and characters of the symmetric groups S_n . Together with work by the English mathematicians, notably Alfred Young, this led to a break through in finite group theory. In early literature, e.g. (Brauer & Nesbitt 1937, Nesbitt 1938, Littlewood 1940), the group algebra $\mathbb{C}[G]$ of a finite group G is synonymously called

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'Frobenius algebra'. A finite group has finite order |G| and one can form the free k-vector space M over (the set underlying) G. The group structure then induces a left and right G action on the bimodule M from the algebra structure of k[G].

Definition 1.1: (regular representations) Let G be a finite group, $x_i \in G$, with multiplication $x_i x_j = \sum_k f_{ij}^k x_k$, with $f_{ij}^k \in \{0,1\} \subset \mathbb{k}$, and multiplication table $[f_{ij}^k]$. We associate the following left 1/right \mathbf{r} representations on ${}_A A$ / A_A and parastrophic matrix $\mathbf{p}_{(a)}$ to $A = \mathbb{k}[G]$.

$$\begin{array}{ll} \mathbf{l}:A \to \operatorname{End}_{\Bbbk}({}_{A}A) & \mathbf{l}_{x_{i}} \cong [f_{i}]_{j}^{k} = [f_{ij}^{k}] & \text{`group matrix'} \\ \mathbf{r}:A \to \operatorname{End}_{\Bbbk}(A_{A}) & \mathbf{r}_{x_{j}} \cong [f_{j}]_{i}^{k} = [f_{ij}^{k}] & \text{`antistrophic matrix'} \\ \mathbf{p}_{(a)}:A \otimes A \to \mathbb{k} & \sum_{k} [f_{ij}^{k}] a_{k} = [(P_{(a)})_{ij}] & \text{`parastrophic matrix'} \ (a_{k} \in \mathbb{k}) \end{array}$$

It is easy to see that \mathbf{l}, \mathbf{r} are, in general reducible, representations induced by left and right multiplication. The parastrophic matrix is not a representation, but is related to a linear form on A. It contains the following important information, solving Frobenius' problem of determining when \mathbf{l} and \mathbf{r} are equivalent:

Theorem 1.2: (Frobenius 1903) If there exist $a_k \in \mathbb{k}$ such that the parastrophic matrix $[P_{(a)}]$ is invertible, then the left and right regular representations are isomorphic ${}_AA \cong A_A$.

Extending to algebras, we have the

Definition 1.3: An algebra A is called Frobenius iff left and right regular representations are isomorphic.

Example 1.4: The reader may check that the commutative polynomial rings $\mathbb{k}[X,Y]/\langle X^2,Y^2\rangle$ and $\mathbb{k}[X]/\langle X^2+1\rangle$ are Frobenius, while $\mathbb{k}[X,Y]/\langle X^2,XY^2,Y^3\rangle$ is not Frobenius. Further examples for Frobenius algebras are the matrix algebras $A=M_n(\mathbb{k})$, where \mathbb{k} is a division ring. In particular for G a finite group $A=\mathbb{C}[G]$ is Frobenius.

1.3. **Finite dimensional Hopf algebras.** Studying the topology of group manifolds Heinz Hopf (1894–1971) introduced in (Hopf 1941) the concept of an 'Umkehrabbildung', that is a comultiplication. The history of Hopf algebras is sketched in (Cartier 2007). We only note that in the old days the term 'Hopf algebra' is what is now called 'Bialgebra'. Coalgebra is the categorical dual notion to algebra, that is we have a vector space C, and two structure maps $\Delta: C \to C \otimes C$, an associative comultiplication, and $\epsilon: C \to I$ a counit, fulfilling the axioms obtained by 'reversing arrows' in the respective diagrams for an algebra (2-1). It is convenient to introduce the Heyneman-Sweedler (Heyneman & Sweedler 1969, Heyneman & Sweedler 1970) index notation for comultiplications. On an element $c \in C$ one sets $\Delta(c) = c_{(1)} \otimes c_{(2)} := \sum_i c_{1i} \otimes c_{2i}$.

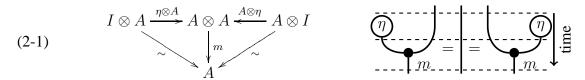
Definition 1.5: A finite dimensional Hopf algebra H is the sextuple $(H, m, \eta, \Delta, \epsilon, S)$ where H is a finite dimensional vector space, m, η are algebra multiplication and unit, Δ, ϵ are comultiplication and counit, and S is the antipode, defined as convolutional inverse of the identity $m(S \otimes Id)\Delta = \eta \epsilon = m(Id \otimes S)\Delta$, fulfilling the compatibility condition: $\Delta(ab) = \Delta(a)\Delta(b)$, see (3-33).

The compatibility relation can be read as 'the comultiplication is an algebra homomorphism' (and *vice versa*). A bialgebra is the above structure without the antipode map. Any graded connected bialgebra is actually a Hopf algebra. We will see below, that a Frobenius algebra

can be described in a similar way using a comultiplication. The Frobenius compatibility law is different, saying that 'the comultiplication respects the module structure' (and *vice versa*). All this will be more obvious when we have the graphical notation available.

2. Graphical calculus

2.1. **History and informal introduction.** We work in a (strict) symmetric monoidal category C, with a tensor as monoidal structure (Majid 1995, Balakov & Kirilov Jr. 2001, Street 2007). Mainly we are interested in the case of $finVect_k$, finite dimensional k-vector spaces, or categories of finite dimensional representations \mathcal{R} , or categories of projective finitely generated left(/right) modules $_R\mathcal{M}$ over a (not necessarily commutative unital) ring R. Category theory (Mac Lane 1971) comes with the diagrammar of commutative diagrams (CDs), where objects are represented as vertices and morphisms as arrows (directed edges) between them. This is one way to define categories, see (Lambek & Scott 1986). Graphical calculus was informally used for a long time, e.g (Brauer 1937). Usually its origin is attributed to Roger Penrose's seminal paper (Penrose 1971). The formal statement that graphical calculus, also called string diagrams, is a sound transformation of category theory is given in (Joyal & Street 1988, Joyal & Street 1991a). A main thrust for developing graphical techniques came from low dimensional topology, that is knot theory (Kauffman 1991, Turaev 1994, Kassel 1995, Ohtsuki 2002) and topological quantum field theory, TQFT, e.g. (Atiyah 1989, Kock 2003). A survey and further literature is in (Seelinger 2011). Graphical calculus is in some sense a (Poincaré) dual picture to commutative diagrams, where morphisms are depicted as labelled vertices (depicted also by boxes called coupons) and objects label the edges connecting them. Such a diagram is called a tangle, it is a representative of an isotopy class of equivalent such diagrams. Every (unoriented) cycle in a commutative diagram gives rise to a tangle equation, which establishes a rewriting rule also called a *move*. For example the unit law for an algebra A in the monoidal category \mathcal{C} has a CD with two triangles, and an equivalent description by two tangle equations



Here we used A in the CD to denote the identity map 1_A , and we dropped the edge label A in the tangles. The tangle was sliced by horizontal dotted lines so that in each slice only one non-identity operation is performed (Morse decomposition). At this time we also need to make clear, that we read tangles downward using the *pessimistic arrow of time* (Oziewicz, talk at ICCA5, Ixtapa, 1999). Also if (nontrivial) crossings occur, we use the *left handed* crossings, see (2-3). The reader needs to exercise caution comparing tangle diagrams as some people are right handed optimists, in that case (2-1) to be read upwards would describe the counit (relabel: $m \mapsto \Delta$, $\eta \mapsto \epsilon$). We find that reversing arrows in a CD, and relabelling them, is equivalent to changing the reading order of the tangle.

Graphical calculus can be interpreted as a 'language' built out of basic *letters*, which form *words* by horizontally (tensoring) or vertically (composition of morphisms) composing them to

form larger tangles. The moves identify different such words into equivalence classes. Sometimes it is convenient to introduce special tangles replacing the coupons depicting them for simplicity and clarity. A selection of graphical entities we are going to use is given as follows:

We may drop identity morphisms and their coupons. We may use a node to depict an algebra multiplication (or comultiplication in the inverted reading of the tangle). We also use the cross for the invertible involutive switch on tensors, and we drop usually the tensor signs on input and output labels, or even the labels if they are clear from the context. Creating and deleting elements depict maps $a:I\to A$ or $f:A\to I$, scalars (ring elements) do not have a graphical counterpart (void diagram) or are depicted by a tangle with no input and output lines (closed graph). A braiding will be depicted by keeping over or under information as usually done in knot theory. This reads as follows:

(2-3)
$$A : A : A : B$$

As in (2-1) we sometimes use labeled circles and not triangles to depict creation and deletion of elements. If we want to distinguish a module A and its duals A^* (or *A) we need either labels or *oriented tangles*. We use downward oriented lines for A and upward oriented lines for A^* (and *A).

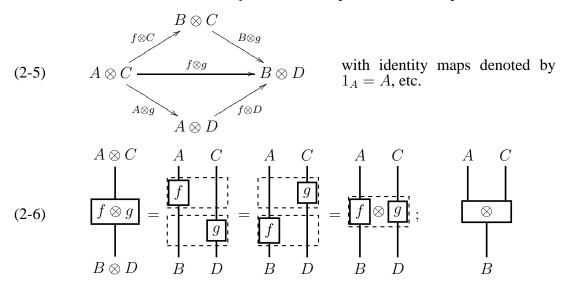
- 2.2. **Basic rules of graphical calculus.** We have not the space to formally introduce graphical calculus in full detail, so we restrict ourselves to present the basic facts how to manipulate tangles.
- 2.3. Horizontal and vertical composition, sliding. We work in a rigid symmetric monoidal category \mathcal{C} , with a 'tensor' bifunctor \otimes as monoidal structure. We have two types of composition of morphisms. The (partial) composition of morphisms in the category is called *vertical composition*, it is depicted as 'stacking' (compatible) coupons of morphisms. We allow rewrites of coupons to be merged at their vertical boundary.

(2-4)
$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$Q \xrightarrow{f} B = Q \xrightarrow{f} C$$

$$C$$

More delicate is the *horizontal composition* of morphisms on tensor products. We have



From this we conclude that we are allowed to i) reduce boxes containing morphisms of the form $f \otimes g$ to two unconnected boxes f and g, and ii) that we are allowed to slide these boxes along each other. Note, that we drop isomorphisms as shown in the rightmost tangle in (2-6). This generalizes in an obvious way to n inputs and m outputs of the tangles.

Warning 2.1: If we work in a braided monoidal category, these moves are no longer available, but need to be altered. Let $R: B \otimes A \to A \otimes B$ be a braiding, acting on elements as $R(b \otimes a) = a_R \otimes b_R = a_r \otimes b_r$ using yet a different Heyneman-Sweedler notation for 2-2-morphisms. The ordinary algebra structure of a symmetric monoidal tensor product on $A \otimes B$ is defined as $(a \otimes b)(c \otimes d) = ac \otimes bd$ or $m_{A \otimes B} = (m_A \otimes m_B)(A \otimes \sigma_{A,B} \otimes B)$ where σ is the switch map on tensors. Now, let $A \#_R B = A \otimes B$ as a k-module, and define a new twisted multiplication, the smash product (recall that $A = 1_A$ etc)

(2-7)
$$m_{A\#_B B} = (m_A \otimes m_B)(A \otimes R_{A,B} \otimes B)$$
 $(a\#b)(c\#d) = ac_R\#b_R d$.

Then the following holds

Theorem 2.2: (Caenepeel, Militaru & Zhu 2002, p. 50) The triple (A, B, R) is a smash product structure if and only if

$$R(b\otimes A)=A\otimes b \qquad \qquad R(B\otimes a)=a\otimes B$$
 (2-8)
$$R(bd\otimes a)=a_{Rr}\otimes b_rd_R \qquad \qquad R(b\otimes ac)=a_Rc_r\otimes b_{Rr}$$
 for all $a,c\in A$ and $b,d\in B$.

The first two identities follow from the unit law, while the second two relate to (strict) associativity. In this case, we get $(f \otimes g) = (f \otimes 1)(1 \otimes g)$ but the other decomposition is not direct $((1 \otimes f)(g \otimes 1) = g_R \otimes f_r)$. Hence 'sliding' fails to be true and needs modification. More generally speaking the smash product is related to the question if $X \cong A \otimes B$ factorizes as an algebra, see loc. cit.. There exists a bijective correspondence between algebra structures on $A \otimes B$ such that the injections i_A, i_B are algebra maps, and smash product structures (A, B, R).

In what follows we mostly use the graphical language for the symmetric monoidal case only, hence assume the sliding move holds.

2.4. Closed structure, isotopy. A tangle is just a representative of an equivalence class of diagrams. The equivalence is isotopy of tangles, which is used in knot theory. We call a tangle an n-m tangle, or tangle with arity (n, m), if it has n input lines (starting at a discrete set of points on a top horizontal line) and m output lines (ending at a discrete set of points on a bottom horizontal line), e.g. a multiplication is a 2-1 tangle, a comultiplication is a 1-2 tangle and scalars will be denoted by 0-0 tangles or even by a void tangle. A *critical point* in a tangle is a point on the plane which has a vertical tangent. We are allowed to smoothly 'bend' lines in a tangle, provided that we keep their topology and do not introduce or destroy critical points or crossings of lines. Moves introducing further freedom to modify tangles impose additional conditions on the underlying category.

In what follows, we need closed structures on the underlying monoidal category. These come in a left and right version. In the case we have a symmetry $\sigma: X \otimes Y \to Y \otimes X$ or a braid, then any two of left duality, right duality, and braiding defines the third. For an in-depth discussion see (Kassel 1995).

Definition 2.3: A monoidal category C is rigid, if for all $X \in C$ there exist X^*, X^* such that the following universal morphisms exists:

• Right duality (often denoted also b_X , d_X):

(2-9)
$$\begin{array}{c} \operatorname{ev}_X: X^* \otimes X \to I_X, \quad \operatorname{cev}_X: I_X \to X \otimes X^*, \\ \operatorname{satisfying} \quad (I_X \otimes \operatorname{cev}_X) (\operatorname{ev}_X \otimes I_X) = I_X \\ (\operatorname{ev}_X \otimes I_{X^*}) (I_{X^*} \otimes \operatorname{cev}_X) = I_{X^*} \end{array} \qquad \begin{array}{c} X^* \quad X \quad \operatorname{cev}_X \\ \operatorname{ev}_X \quad X^* \end{array}$$

• Left duality (often denoted also \tilde{b}_X , \tilde{d}_X):

(2-10)
$$\begin{array}{c} X \text{ev} : X \otimes X^* \to I_X, \quad _X \text{cev} : I_X \to X^* \otimes X, \\ \text{satisfying} \quad (I_{^*X} \otimes _X \text{cev})(_X \text{ev} \otimes I_{^*X}) = I_{^*X} \\ (_X \text{ev} \otimes I_X)(I_X \otimes _X \text{cev}) = I_X \end{array} \quad \begin{array}{c} X \quad ^*X \quad _X \text{cev} \\ X \text{ev} \quad ^*X \quad X \end{array}$$

• Symmetry (braiding) $\sigma_{X,Y}:X\otimes Y\to Y\otimes X$, satisfying the braid equation and invertibility

$$(2-11) \qquad (\sigma \otimes I)(I \otimes \sigma)(\sigma \otimes I) = (\sigma \otimes I)(\sigma \otimes I)(I \otimes \sigma); \qquad \sigma \sigma^{-1} = I \otimes I$$

The last equation as tangles represent the Reidemeister 3 move **R3** (2-14) and Reidemeister 2 move **R2** (2-15).

The conditions on \mathbf{ev} , \mathbf{cev} are depicted as $topological\ move$ (or Reidemeister 0 move $\mathbf{R0}$), that is it allows deletion or introduction of two compatible extrema. This move is also ambiguously addressed as 'yanking', but we will see below that Frobenius bilinear forms also allow a 'yanking' of lines, so we reject this term. We have introduced oriented lines, to depict objects X, X^* and X. If the distinction between left and right dual is vital, we need to apply labels. The topological move $\mathbf{R0}_r$ for the right duality (the left dual tangles are obtained by inverting

orientation) depicts the conditions in (2-9) (and with inverted orientation that of (2-10)).

(2-12)
$$\mathbf{R0}_r$$
:

The Reidemeister 2 move **R2** depends on what is encoded by the lines in a tangle, see section 2.5. If the lines are assumed to be one dimensional 'strings' (*sic* the name of the calculus), then straightening a loop introducing a twist θ in a string does not matter. In this case the Reidemeister 2 move is given by (graphically as lhs of (2-15))

$$\theta := (I \otimes ev)(\sigma \otimes I)(I \otimes cev) = I$$

If we assume that lines in a tangle are more complicated objects, e.g. ribbons or cylinders (in TQFT, see sec. 2.5), then we need to keep track of the twists. In this case $\theta \neq I$, and the Reidemeister 2 move needs another loop θ^{-1} with an inverse braid to compensate.

To summarize, we have the following isotopy moves on tangles

where the braiding is trivial for the switch map σ . The two alternative Reidemeister 2 moves (with and without a twist/braiding) read

In the symmetric monoidal case the twist morphism θ_A can be seen roughly as the composition $A \to {}^*A \to ({}^*A)^*$. If ${}^*A \cong A^*$ then θ_A is the canonical identification $A \cong (A^*)^*$, explaining why loops can be undone. The modified Reidemeister 2' move equals identity using the two identifications $A \cong (A^*)^*$ and $A \cong {}^*({}^*A)$ explaining why two loops are necessary, and the braiding of course.

Moreover, we can move lines above and below extrema (evaluation and coevaluations). As an example look at

$$(2-16) \qquad \qquad = \qquad \qquad ; \text{ with } \qquad := \qquad \vdots$$

The other cases are similar.

To relate right and left duality, we need the braiding. If $\sigma^2 = I \otimes I$, is the ordinary switch, we get a symmetric monoidal category and can drop the left/right distinction, as they are related by the identity morphism, see left tangle equation in (2-17). If σ is a proper braid (2-11), then we can derive the left duality from the right duality. This is shown in the following equations for the 'cup' tangles ev

The 'cap' tangles **cev** are related in a similar fashion. There are some subtleties in the interplay between dualities and twists, which we do not contemplate here further, see (Joyal & Street 1991*b*, Turaev 1994, Kassel 1995, Street 2007).

In a symmetric monoidal category we can assume **R0**, **R1**, **R2**, **R3** with a trivial twist. As we also get ${}^*A \cong A^*$ we can drop orientation too. This is called 'ambient isotopy'. If we work in a ribbon category, we replace **R2** by its modified version **R2**' and keep track of twist and braid morphisms, this is called 'regular isotopy'; for terminology see (Kauffman 1991). In what follows we will for simplicity work mainly in the symmetric monoidal setting.

2.5. Tangles not depicting strings. 'String diagrams', related to symmetric monoidal categories, take their name from picturing them literally as strings, assumed to have zero radial extension. Such strings cannot be twisted (or twisting them is irrelevant). However, there are mathematical structures which are sensitive to twisting. They may be depicted e.g. by ribbons, and lead accordingly to 'ribbon categories'. This is a reason to speak instead about 'tangle diagrams'. However, there are still more thickenings of tangles, such as cylinders in topological quantum field theory (TQFT). If we still use 'strings' to depict them, we are forced to adjust the allowed moves, e.g. change the Reidemeister 2 move. This leads to different notions of isotopy, see (Kauffman 1991). Here we just depict ribbons and the relevant tangles for TQFT for reference, but in the sequel we will just use strings which reduces the artwork considerably and does

not loose information if care is taken about using only allowed rewriting rules.

$$(2-18) \qquad \qquad = \qquad ; \qquad \bigcirc; \qquad \bigcirc; \qquad ; \qquad \bigcirc$$

The left equation shows how a loop, if straightened, produces a 2π twist in the ribbon. The three right most diagrams depict two dimensional surfaces, cobordisms, which connect the (oriented, one dimensional) circles at the top with (oriented) circles at the bottom. They are called 'disk' (the unit, or counit if inverted), 'trinion' (the Frobenius algebra product, or coproduct if inverted), and the 'cylinder' (identity map). We have depicted a twisted line on the cylinder, showing that one can also have a twist (Dehn twist) on cylinders. A TQFT based on these diagrams allows all two dimensional Riemannian surfaces to be constructed and characterized, see (Lawrence 1996).

3. Frobenius and Hopf Algebras

In this section we recall some facts about, and most importantly some characterizations of, Frobenius algebras. We want to emphasize especially the difference between the multiplications in endormorphisms rings $\operatorname{End}_R(M)$ over an R-module M and the left (right) action induced by an algebra multiplications $m_A: A\otimes A\to A$, where we look at the right (left) factor A as a left A-module A (right A-module A). Our main sources are (Yamagata 1996, Kadison 1999, Caenepeel et al. 2002, Murray 2005, Lorenz 2011, Lorenz & Fitzgerald Tokoly 2010). General texts on Hopf algebras and modules are (Sweedler 1969, Abe 1980, Kasch 1982, Caenepeel 1998, Street 2007).

Usually we work over a field, but we will recall here some more general notions where we also allow more general base rings R, examples being a residual field, a ring extension or even a noncommutative ring.

3.1. **Actions, coactions, representations and two multiplications.** To understand the similarities and differences between the Frobenius and endormorphism structures, we need to look briefly at algebra representations. We do that superficially only to the extent which is necessary for our purpose.

Let ${}_RM_S$ be a (p.f.) R,S-bimodule and $\mathcal{E}(M)=\operatorname{End}(M)\cong M\otimes_S M^*$ be the endormophism ring over M. Let $\{x_i\}_{i=1}^n$ be a set of generators for M and $\{f_i\}_{i=1}^n$ be a dual basis for M^* , i.e. $\operatorname{ev}(f_i\otimes x_j)=f_i(x_j)=\delta_{ij}$. Given a nondegenerate bilinear form $\beta:M\otimes M\to R$ with inverse $\overline{\beta}=\sum x_i\otimes y_i$ (see (3-27)). The bilinear form provides us with another set of generators $\{y_i\}_{i=1}^n$ for M, such that $\sum \beta(m,x_i)y_i=m$ for all $m\in M$. For simplicity we denote $\beta^*=\overline{\beta}$, as it is distinguished by its type signature, see (3-2). The next tangles describe a left A action on ${}_RM_S$, and how left-duality allows to define therefrom a right A action on ${}_SM_R^*$. For the moment we use 2-tangles, where the area depicts the ring in question, for more details on such tangles see e.g. (Luauda 2006, Khovanov 2010). The rightmost tangle is a coaction for

which similar results hold by tangle symmetry.

(3-1)
$$\begin{array}{c} A_R M_S & sM_R^* A & sM_R^* A \\ R \downarrow R \downarrow S ; & S \downarrow R \mid R \\ R \downarrow S & SM_R^* & SM_R^* & SM_R^* & A_R M_S \end{array}$$

The Frobenius property induces an isomorphism between left modules ${}_AM$ and right modules ${}_{A}M$, which does not follow from duality alone. We can use the bilinear from to define a right action on M from the right action on M^* as follows

We will study the properties of this isomorphism below in more detail.

A (finite) algebra A can be represented by a map into an endomorphism ring $\operatorname{End}(M) \cong M \otimes M^*$. The algebra product is mapped homomorphically onto the natural product of endomorphisms given by the universal evaluation map, that is composition of endomorphisms. For our purpose we use maps h, y, see (3-3), such that $y \circ h = 1_A$, that is we use faithful representations. In the light of Wedderburn's theorem we may even assume, for simplicity, that $A \cong \operatorname{End}(M)$, hence assuming A is simple such that $h \circ y = 1_{\operatorname{End}(M)}$. Now we can look 'inside' the multiplication in A obtaining the left isomorphism in the next display.

$$(3-3) \qquad \qquad \cong \qquad \stackrel{\text{h}}{ } \qquad \stackrel{$$

The second isomorphism is more subtle, as it involves the bilinear form. This multiplication is called β -multiplication and operates on $M \otimes M$. The choice of β has to be compatible with the morphisms \sqcup, \sqcap in (3-3). With $a,b \in A$ such that $a = \sum a_{ij}x_i \otimes y_j, \ b = \sum b_{ij}u_i \otimes v_j$ one obtains the multiplication

(3-4)
$$ab = \sum a_{ij}x_i \otimes y_j \sum b_{ij}u_i \otimes v_j = \sum a_{ij}b_{lm}\beta(y_j, u_l)x_i \otimes v_m$$

We remark here only, that the information flow in the endomorphism ring situation is different (having upwards/back in time flow) compared to the β -multiplication (related to a Frobenius algebra) which has only downward information flow. This difference allows one in quantum

teleportation to *choose* an entangled Bell state (related to $\overline{\beta}$) and make different Bell measurements (related to β up to unitary transformations), while the endomorphic situation does not allow one this freedom. Also in linguistic models of meaning this may have some implications, see subsection 3.10.

3.2. Some notions from ring and module theory. All modules we are going to use are finitely generated projective (f.p.) over a base field or ring. This is implied by the invertibility of the Frobenius bilinear form (parastrophic matrix) hinging on a good duality theory. This enables one to dualize algebra structures providing a coalgebra structure, which fails in the general situation. Let A be an R-algebra with structure maps m_A, η_A . We denote by A^{op} the opposite algebra over the same R-module A, with the opposite multiplication $m_A^{op} = m_A \circ \sigma$. It is useful to introduce the *enveloping algebra* $A^e = A \otimes A^{op}$, which allows one to rewrite A, A-bimodules AM_A as A^e -left modules.

A derivation $D:A\to M$ is a linear operator from the A,A-bimodule A to the A,A-bimodule M, such that

(3-5)
$$D(ab) = D(a).b + a.D(b) + D + D + D + D$$

where the module M is represented by a bold line. The bold-unbold 'multiplication' like tangle is the right/left action of A on M. Let $\mathrm{Der}_R(A,M)$ be the R-module of derivations. A derivation D_m is called *inner derivation* if there exists an $m \in M$ such that $D_m(a) = am - ma$. Now define the space of A-invariants of M as $M^A := \{m \in M \mid am = ma\}$. It is obvious that for all $m \in M^A$ the inner derivation vanishes $D_m = 0$. If M = A as R-modules, the space of invariants is just the kernel of the multiplication map $I(A) = \mathrm{Ker}(m_A)$. One finds the following sequence to be exact

$$(3-6) 0 \to M^A \to M \to \operatorname{Der}_R(A, M)$$

Using the isomorphisms between A, A-bimodules and A^e -left modules shows that $M^A \cong \operatorname{Hom}_{A^e}(A, M)$ and $M \cong \operatorname{Hom}_{A^e}(A^e, M)$. It is also easy to see that $m_A : A^e \to A$ is an epimorphism. Hence the following sequence is exact

(3-7)
$$0 \to I(A) = \operatorname{Ker}(m_a) \to A \otimes A^{op} \to A \to 0$$

A situation which is important in the Frobenius case is when this sequences is split. That is there exists a map $\delta:A\to I(A)::a\mapsto\delta(a)=a\otimes 1-1\otimes a$ whose image I(A) in A is an ideal AI(A)=I(A)=I(A)A. Then $A^e=A\otimes A^{op}$ decomposes as a direct sum $A^e=I(A)\oplus A$ and there is an idempotent pair (e,1-e) projecting onto the two spaces. This gives by standard algebra arguments some structure results.

Lemma 3.1:

(3-8)
$$\operatorname{Hom}_{A^e}(I(A), M) \cong \operatorname{Der}_R(A, M)$$

Applying the functor $\operatorname{Hom}_{A^e}(-,A)$ to the exact sequence (3-7) shows that $HH^1(A,M)=\operatorname{Ext}_{A^e}^1(A,M)\cong\operatorname{Der}_R(A,M)/\operatorname{Inn}\operatorname{Der}_R(A,M)$, where HH^1 is the first Hochschild cohomology group. As an aside, having a Hopf algebra structure allows one to formalize several cohomology theories in a uniform manner, see (Sweedler 1968). The graphical calculus is not (very) sensitive to the underlying ring structure, so we do not go deeper into ring theory here. The main result we quote, establishes the existence of an spiting idempotent for the class of finite projective algebras we are interested in.

Theorem 3.2: Let R be a commutative ring. For R-algebras A the following statements are equivalent:

- (i) A is projective as a left A^e -module.
- (ii) The exact sequence (3-7) for A^e -modules is split.
- (iii) There exists a splitting idempotent element $e = \sum e_{(1)} \otimes e_{(2)} \in A \otimes A$ such that for all $a \in A$, ae = ea and $\sum e_{(1)}e_{(2)} = 1$ holds.

An R-algebra A is separable iff A/R is a separable ring extension. That is $m:A^e\to A$ is a split epimorphism of A^e -modules. By the above theorem this is equivalent to say that A is A^e -projective or that there exists a splitting idempotent e as in (iii). For further details see (Kadison 1999, Sec. 5.2). The conditions in (iii) translates into the following graphical statements ($\eta:I\to A$ is the unit map)

$$(3-9) \qquad \qquad \bigcap_{\sum e_{(1)} \otimes e_{(2)}} : \qquad \bigcirc = \bigcirc = ; \bigcirc = \bigcirc$$

We defined the comultiplication map $\delta:A\to A\otimes A::a\mapsto ae=ea$. Coassociativity follows from the symmetric definition of δ and from associativity of the product in A and the 'sliding' of morphisms. Try it! For more information about splitting idempotents and quadratic algebras see (Hahn 1994, Caenepeel et al. 2002), as we want to avoid to discuss Azumaya and Taylor-Azumaya algebras. We will use generators and bases so we quote two more standard results from algebra, guaranteeing the existence of bases (generators).

Theorem 3.3: Any projective separable algebra A over a commutative ring R is finitely generated. A separable algebra A over a field \mathbb{k} is semisimple.

Using this theorem, we find a finitely generated projective R-module M, with generators $\{x_i\}_1^n$ and a dual module M^* with dual basis $\{f_i\}_1^n$ such that $A \cong M \otimes_R M^*$. The Frobenius homomorphism will allow us to replace the dual module A^* by A and the dual basis by a reciprocal basis $\{y_i\}_1^n$.

3.3. **Frobenius functors.** Let A, S be rings and let ${}_{A}\mathcal{M}$, ${}_{S}\mathcal{M}$ be the categories of (f.p.) A-modules and S-modules. Let $i: S \to A$ be an injection, then any A-module can be turned into an S module. This defines the restriction functor R in the opposite direction

(3-10)
$$R: {}_{A}\mathcal{M} \to {}_{S}\mathcal{M} :: {}_{A}M \mapsto {}_{S}M :: sm = i(s)m$$

This is an instance of a change of base functor. Now R has a left adjoint $T \dashv R$ (induction functor) and a right adjoint $R \dashv H$ (coinduction functor) defined as follows.

(3-11)
$$T: {}_{S}\mathcal{M} \to {}_{A}\mathcal{M} :: \left\{ \begin{array}{l} {}_{S}M \mapsto A \otimes_{S} {}_{S}M \\ f \mapsto A \otimes_{S} f \end{array} \right.$$

$$H: {}_{S}\mathcal{M} \to {}_{A}\mathcal{M} :: \left\{ \begin{array}{l} {}_{S}M \mapsto \operatorname{Hom}_{S}(A, {}_{S}\mathcal{M}) \ni h \\ f \mapsto f \circ h \end{array} \right.$$

This means one has $\operatorname{Hom}_A(\mathsf{T}(M),N) \cong \operatorname{Hom}_S(M,\mathsf{R}(N))$ and $\operatorname{Hom}_S(\mathsf{R}(N),M) \cong \operatorname{Hom}(N,\mathsf{H}(M))$. The Frobenius property is captured by the

Definition 3.4: A ring extension A/S is a Frobenius extension if and only if H and T are naturally equivalent as functors from ${}_{S}\mathcal{M} \to {}_{A}\mathcal{M}$.

We get a Frobenius structure, see (Kadison 1999, Caenepeel et al. 2002, Khovanov 2010), that is a triple $(\beta, \{x_i\}, \{y_i\})$ where $\lambda \in \operatorname{Hom}_{S-S}(A, S)$ is the Frobenius homomorphism with $\beta(a,b) = \lambda(ab)$ (inverse $\overline{\beta}: S \to A \otimes A$), and $\{x_i\}, \{y_i\}$ are generators of A fulfilling the β -multiplication equations (see tangles (3-27) and (3-28)).

(3-12)
$$\sum_{i} x_i \beta(y_i, a) = a, \qquad \sum_{i} \beta(a, x_i) y_i = a$$

This is a generalization of the theorem 3.6, valid for noncommutative rings, with many interesting applications, see for example (Khovanov 2006). Pairs of functors (T, H) with $T \dashv R \dashv H$ such that $T \cong H$ is an isomorphism are called Frobenius pairs of Frobenius functors.

3.4. Graphical characterization of Frobenius algebras. Let $_R\mathcal{M}$ be a tensor category of (f.p.) R-modules, and $A \in _R\mathcal{M}$. Let A be an R-algebra with structure maps (μ_A, η_A). If A is Frobenius then further structure maps exist, such as the Frobenius homomorphism $\lambda: A \to R$, or equivalently an associative bilinear form (see section 3.7) $\beta = \lambda \circ \mu_A: A \otimes A \to R$. However, it is graphically more effective to use the dual Λ of the Frobenius homomorphism and the splitting idempotent element e to define a coalgebra structure $\delta: A \to A \otimes A$, as in (3-9). we define the coproduct as follows

$$(3-13) \qquad \qquad \delta_A := \boxed{\overline{\beta}} m_A = \boxed{\overline{\beta}} m_A = \boxed{\overline{\beta}}; \qquad \boxed{\overline{\beta}} = \boxed{\Lambda}$$

It is easy to show graphically that Λ is a unit for m_A and λ is a counit for δ_A . Hence we arrive at the

Definition 3.5: A Frobenius algebra A is a quintuple $(A, m_A, \lambda, \delta_A, \Lambda)$ such that λ is a counit for δ_A , Λ is a unit for m_A and the multiplication and comultiplication fulfill the compatibility law

$$(3-14) \qquad \qquad = \qquad = \qquad = \qquad$$

together with the units and 'yanking' rules given in (3-13) and their duals.

If we interpret multiplication as an A action of the (left/right) A-module A, and comultiplication similarly as (left/right) coaction, then this compatibility relation reads '(left/right) actions and coactions commute'.

3.5. **Algebraic characterizations of Frobenius algebras.** Frobenius algebras can be characterized in a number of ways, emphasizing different aspects of this structure.

Theorem 3.6: (Caenepeel et al. 2002) Let A be an n dimensional k-algebra, the following statements are equivalent:

- (1) A is Frobenius.
- (2) There exists a *Frobenius isomorphism* $\beta^r \in \operatorname{Hom}_{\Bbbk}({}_AA, {}_AA^*)$ for left A-modules in ${}_A\mathcal{M}$.
- (3) There exists a *Frobenius isomorphism* $\beta^l \in \operatorname{Hom}_{\mathbb{k}}(A_A, A_A^*)$ for right A-modules in \mathcal{M}_A .
- (4) The left regular representation 1 and right regular representation r are equivalent.
- (5) There exists $a_i \in \mathbb{k}$ such that the parastrophic matrix (1.1) is invertible.
- (6) There exists a nondegenerate associative bilinear form $\beta: A \times A \to \mathbb{k}$ (associativity: $\beta(ab,c) = \beta(a,bc)$ for all $a,b,c \in A$).
- (7) There exists a hyperplane in A that does not contain a nonzero right ideal of A.
- (8) There exists a pair $(\lambda, \overline{\beta})$, called Frobenius pair, where $\lambda \in A^*$ is a *Frobenius homomorphism* and $\overline{\beta} : \mathbb{k} \to A \otimes A$ $(\overline{\beta} = \Delta(1) = \sum \overline{\beta}_{(1)} \otimes \overline{\beta}_{(2)})$ such that for all $a \in A$

$$(3-15) a\overline{\beta} = \overline{\beta}a \sum \lambda(\overline{\beta}_{(1)})\overline{\beta}_{(2)} = \sum \overline{\beta}_{(1)}\lambda(\overline{\beta}_{(2)}) = 1$$

where we used Heyneman-Sweedler notation for the comultiplication.

3.6. **Some properties of Frobenius and Hopf algebras.** Frobenius algebras share some similarities with Hopf algebras, but also exhibit different features. We will discuss the relation between Frobenius and finite Hopf algebras in section 3.9, moreover see the chapters of (Majid 2012, Vercruysse 2012) in this book.

In quantum information theory it is appropriate to distinguish two extremal cases.

Definition 3.7: A Frobenius algebra A is called *special* or *trivially connected*, if the loop operator equals identity $l=m_A\circ\delta_A=1_A$ (see (3-17)), or connected if l is invertible. A Frobenius algebra is called *totally disconnected* if the loop operator decomposes as $l=m_A\circ\delta_A=\Lambda\circ\lambda$.

The special or connected case (l.h.s. eqn. (3-16)) shows that by 'yanking' one generates a tensor state which cannot be factored, while the (r.h.s) equation shows that the totally disconnected

case produces a product state.

$$(3-16) \qquad \qquad \overline{\beta} \qquad \overline{\beta$$

In (Coecke, Pavlovic & Vicary 2012, Coecke, Paquette & Pendrix 2008) it is shown that special Frobenius algebras are in one to one correspondence with a choice of an orthonormal basis. Moreover one can characterize classical structures and complementarity on q-bits using special Frobenius algebras (Coecke & Duncan 2011).

(3-17)
$$l = \beta$$
; $f = \beta$; spider thm.

Let $l = m \circ \delta$ be a 'loop'. One has the following normal form (or spider) theorem

Theorem 3.8: Let A be a symmetric Frobenius algebra, that is $m_A^{op} = m_A \circ \sigma = m_A$, then any tangle 'foo' with arity (n,m) (n inputs, m-outputs) can be transformed using Frobenius moves and associativity to the normal form $\delta^{m-1} \circ l^r \circ m^{n-1}$ (with $m^0 = l = \delta^0 = \operatorname{Id}$) for some non-negative integer r, see (3-17). If A is special $l = 1_A$.

This theorem is proven by recursion. The Frobenius property (3-14) and associativity for m_A and δ_a allows us to interchange the order of multiplications and comultiplications. In this way the inputs can be multiplied together, and the outputs produced by comultiplications. In this process a certain number of 'loops' occur, which vanish if A is special. Composing an n, m-tangle with k loops in its normal form with an m, p-tangle with l loops produces a n, p-tangle with k+l+m loops. If A is not symmetric one needs to deal with the Nakayama automorphisms α , see section 3.8. We note here that α can be constucted using the bilinear forms $\beta, \overline{\beta}$ and the trivial symmetry σ (switch) as in (3-17) along the lines we constructed the twist θ in (2-15) from left/right dualities and the braiding.

Theorem 3.9: A special symmetric Frobenius algebra A is a bialgebra.

Another interesting property, which can be used e.g. in singular value decomposition (Fauser 2006), is the following fact, true for any convolution algebra $\operatorname{Hom}(A,A)$, hence for Hopf and Frobenius.

Theorem 3.10: The operators 'loop' $l = m_A \circ \delta_A$ of arity (1,1) and 'fork' $f = \delta_A \circ m_A$ of arity (2,2), see (3-17), fulfill the same minimal polynomial, hence have the same positive spectrum up to a null-space.

In the case of a special Frobenius algebra we see that $l=1_A$ and hence f is a projector onto a space isomorphic to A in $A\otimes A$.

Kuperberg ladders (Kuperberg 1991, Fauser 2002) are the counterparts in a Hopf algebra to the leftmost/rightmost tangles in (3-14). A Hopf algebra comes with an antipode S ((1.5) and sec. 3.9 below) causing the ladder tangle to be invertible (lhs in (3-19)). These tangles play a role in invariant theory of 3-manifolds.

The rightmost equation in (3-19) shows a Frobenius algebra homomorphism $H:A\to B$ such that $m_B\circ (H\otimes H)=H\circ m_A$. In red/green-calculus the map in use is the Hadamard gate, which is invertible, allowing a 'color change', that is an change of algebra structure. As special Frobenius algebras encode bases, this is essentially an entangling operator changing the underlying classical structure. In the Hopf algebraic case Sweedler developed a powerful cohomology theory (Sweedler 1968) with a Hopf algebra action, which provides a classification of such maps. Algebra homomorphisms fall into the trivial cohomology class.

As a last example in this subsection we consider a module A carrying a Hopf and a Frobenius algebra structure at the same time. To make this situation well behaved we demand the following distributive laws (also called Laplace property (Rota & Stein 1994)) to hold as compatibility relations. (White dots belong to the Hopf algebra, black to Frobenius.)

$$(3-20) \qquad \qquad = \qquad =$$

In (Fauser 2001, Brouder, Fauser, Frabetti & Oeckl 2004), see also (Carroll 2005), it was demonstrated how this 'Laplace Hopf algebras' produce via twistings all multiplicative structures in a perturbative quantum field theory. See also (Fauser & Jarvis 2006, Fauser 2008) for a fancy number theoretic application. In (Fauser & Jarvis 2004, Fauser, Jarvis, King & Wybourne 2006, Fauser, Jarvis & King 2010) among others it was demonstrated how this structure underlies invariant rings, providing powerful tools to shorten proofs and allowing to solve otherwise difficult problems. As a mental picture, the Hopf algebra operates on a tensor module as 'concatenation',

while the Frobenius structure encapsulates the discrete permutation symmetries S_k on tensor terms $\sigma: \otimes^k A \to \otimes^k A$.

3.7. **Associative regular bilinear forms and Frobenius homomorphisms.** Frobenius algebras come with a second way, beside the closed structures, to identify f.p. modules with their duals. This is essentially using the parastrophic matrix from (1.1).

Definition 3.11: (Murray 2005) Let \mathbb{k} be a residual field. A *regular associative bilinear form* is a \mathbb{k} -linear map $\beta \in \mathsf{Bil}^r_{ass}(A,\mathbb{k}) : A \otimes_{\mathbb{k}} A \to \mathbb{k}$ such that

$$\beta(ab,c)=\beta(a,bc)$$
 associativity

(3-21)
$$\forall a \in A \text{ with } a \neq 0, \exists b \in A \text{ such that } \beta(a,b) \neq 0$$
 non degenerate

The linear form $\lambda := \beta(-,1) = \beta(1,-)$ is called *Frobenius homomorphism*. If $\lambda(ab) = \lambda(ba)$, that is the Nakayama automorphisms $\alpha = \operatorname{Id}$ (see section 3.8), it is called *trace form* (see tangle in (3-22)).

Usually bilinear forms are depicted as cup-tangles, to avoid confusion with the evaluation maps from the closed structure we denote them as coupons.

$$(3-22) m_A \beta = \beta m_A; \beta \beta = \beta \beta; \beta \beta; \beta \beta \beta$$

Two bilinear forms are related by a *homothety*, $\beta \simeq \beta'$ if there exists a unit $k \in \mathbb{k}^{\times}$ and an automorphism $V \in \operatorname{Aut}_{\mathbb{k}}(A)$ such that

$$\beta(a,b) = k\beta'(Va, Vb)$$

Two bilinear forms related by a homothety are not essentially different and the set Bil_{ass}^r can be partitioned into homothety equivalence classes. The rightmost equation in (3-22) shows further that the Frobenius homomorphisms λ and $\lambda' = k\lambda V$ are related if V is an algebra homomorphism, as in the rhs of (3-19).

3.8. Nakayama automorphism. We call a bilinear form symmetric if $\forall a,b \in A$. $\beta(a,b) = \beta(b,a)$. Note that this does not imply that A is symmetric, in general $A \neq A^{op}$. The Nakayama automorphism $\alpha \in \operatorname{Aut}_{\Bbbk-alg}(A)$ measures the deviation from symmetry of β .

$$\beta(a,b) = \beta(b,\alpha(a))$$

The Nakayama automorphism is unique up to inner automorphisms. First fix an isomorphism $\phi: \operatorname{Hom}({}_AA, {}_AA^*)$. Any other such isomorphism is given by first applying an automorphism to ${}_AA$ and then applying ϕ . This results in a transformation to a new Frobenius linear form $\lambda' = u\lambda: c \mapsto \lambda(ck)$ and a new bilinear form $\beta'(a,b) = \beta(a,bu)$, where $u \in A^\times$ is a unit in the algebra A. The corresponding Nakayama automorphism transforms as $\alpha' = I_u \circ \alpha$, where $I_u(a) = uau^{-1}$ is an inner automorphism.

The bilinear form β is symmetric iff $\alpha = \text{Id}$. Note that in a braided setting the ordinary switch can be addressed as a virtual crossing (Kauffman 1999) not encoding over/under information.

The nondegenerate bilinear form $\beta: A \otimes A \to \mathbb{R}$ induces the notion of an adjoint on $\operatorname{End}(A)$ via $\beta(a, V^t b) := \beta(V a, b)$, which we call *transposition*. This transposition fulfills the usual properties such as linearity, $(UV)^t = V^t U^t$, and $(U^{-1})^t = (U^t)^{-1}$. In the presence of a nontrivial Nakayama automorphism transposition is in general *not* an involution.

(3-26)
$$\beta(a, V^{t^2}b) = \beta(V^t a, b) = \beta(\alpha^{-1}b, V^t a) = \beta(V\alpha^{-1}b, a) = \beta(a, \alpha V\alpha^{-1}b),$$

hence we get $V^{t^2}=\alpha V\alpha^{-1}$. If the Frobenius algebra is symmetric, then $\alpha=\mathrm{Id}$ and transposition is an involution.

Lemma 3.12: (Murray 2005) Bilinear forms β and β' are homothetic iff $\exists k \in \mathbb{k}^{\times}$ and $V \in \operatorname{Aut}_{\mathbb{k}}(A)$ such that $\rho_u = kV^tV \in \operatorname{Aut}_{\mathbb{k}}(A)$.

The reader may compare ρ_u to the definition of a positive operator in quantum mechanics. In the same line of thought, we notice the following: Let A be a Frobenius \mathbbm{k} -algebra with bilinear form β and Nakayama automorphism α . As seen above, two homothetic forms $\beta'(a,b)=\beta(a,bu)$ are related by a unit $u\in A^\times$ with Nakayama's $\alpha'=I_u\circ\alpha$. This shows that the order of the Nakayama automorphism is independent of the choice of the form in the homothety equivalence class. This can be used to define the following norm function: Let $\alpha^n=1$ be a Nakayama automorphism of finite order, define the algebra norm $N_\alpha(a):=a\alpha(a)\ldots\alpha^{n-1}(a)$. This norm can be interpreted as the evaluation of a term t^n at a in the Ore ring of right twisted polynomials $A[t,\alpha]$, see (Lam & Leroy 1988, Bueso, Gómes-Torrecillas & Verschoren 2003, Abramov, Le & Li 2005). If $\alpha^n=I_a$ then $\alpha(a)=a$ and one gets $\alpha''=a$ 0 and $\alpha''=a$ 1 in the involutive case, important for quadratic algebras and Clifford algebras (Hahn 1994), or more generally for *-algebras, one defines such norms using 'special elements' or directly using the involution. In quantum theory one usually assumes an involutive *-automorphism.

The closed structures allow us to relate morphisms f in $\operatorname{Hom}(A,B)$ to dualized morphisms f^* in $\operatorname{Hom}(B^*,A^*)$ etc. bending lines up or down. That is using the topological move R0 (2-12) or 'yanking'. Having the Frobenius bilinear form β available, we have a second possibility to bend lines, which this time produces maps in $\operatorname{Hom}(A,B^*)$ etc. We need first to define the inverse map $\overline{\beta}: \mathbb{k} \to A \otimes A$, which exists due to nondegeneracy of β . We should write β_A and $\overline{\beta}_A$ for the components of the map on the category, but to unclutter notation we drop these indices.

$$(3-27) A \xrightarrow{A \otimes \overline{\beta}} A \otimes A \otimes A \xrightarrow{\beta \otimes A} A \downarrow \qquad \qquad \downarrow \overline{\beta} \qquad = \qquad \downarrow \beta$$

 $\overline{\beta}$ is a right inverse. Using the Nakayama automorphism α we see that it is also a left inverse $(\overline{\alpha} = \alpha^{-1})$:

$$(3-28) \qquad \qquad \downarrow = \underbrace{\frac{\overline{\beta}}{\beta}}_{\beta} = \underbrace{\frac{\overline{\alpha}}{\alpha}}_{\beta} = \underbrace{\frac{\overline{\beta}}{\overline{\beta}}}_{\beta} = \underbrace{\frac{\overline{\beta}}{\beta}}_{\beta}$$

Here the crossings are 'virtual' that is the switch σ . The moves in (3-27) and (3-28) should be compared with the topological moves (2-12) for the the cup/cap tangles of the closed structure. Here the left/right aspect is taken care of by the Nakayama automorphism.

A main characterization of a Frobenius algebra in theorem 3.6, and one which generalizes, is given by the Frobenius isomorphism ${}_AA\cong {}_AA^*$ of the left A-modules. The Frobenius bilinear form and its dual allow us to construct such morphisms together with the closed structures (Fuchs 2006, Fuchs & Stigner 2008). We define left/right module maps $\beta^r\in \operatorname{Hom}(A,A^*)$, $\beta^l\in \operatorname{Hom}(A,{}^*A)$ and their inverses. The left/right aspect refers to the closed structures involved.

$$(3-29) \qquad \begin{array}{c} A & A \\ \downarrow \\ \beta^r \end{array} := \begin{array}{c} A^* \\ \beta \\ A^* \end{array} ; \begin{array}{c} A^* \\ \overline{\beta}^r \end{array} := \begin{array}{c} A^* \\ \overline{\beta}^r \end{array} ; \begin{array}{c} A^* \\ \overline{\beta}^r \end{array} = \begin{array}{c} A^* \\ \overline{\beta}^r \end{array}$$

and

$$(3-30) \qquad \begin{array}{c} A \\ \downarrow \\ \beta^{l} \end{array} := \begin{array}{c} {}^{*}A \\ \beta \end{array} ; \begin{array}{c} {}^{*}A \\ \overline{\beta}^{l} \end{array} := \begin{array}{c} {}^{*}A \\ \overline{\beta}^{l} \end{array} := \begin{array}{c} {}^{*}A \\ \overline{\beta}^{l} \end{array} = \begin{array}{c} {}^{*}A \\ \overline{\beta}^{l} \end{array}$$

The proof that the identity holds in (3-29) and (3-30) requires *both* the inverse Frobenius bilinear form (3-28) and the topological move for right/left duality (2-12), and is left as an easy exercise.

We close this section about the Nakayama automorphism and its implications on 'yanking' moves by showing that the left/right duality imposed by the closed structures is related to the

module homomorphisms β^{\bullet} and $\overline{\beta}^{\bullet}$ (with $\bullet = r$ or l, Frobenius isomorphisms)

In (Fuchs 2006) it is further graphically shown, that the Nakayama automorphism $\alpha = \overline{\beta}^l \beta^r$ is actually an algebra automorphism. Compare this form of α with the form for α given in (3-17) which does not use the closed structure or the braid.

3.9. **Finite Hopf algebras as Frobenius algebras.** Hopf algebras will be discussed at length in other chapters of this book (Majid 2012, Vercruysse 2012). We will provide here only the basic facts which relates them to Frobenius algebras. In definition 1.5 we saw that a Hopf algebra over H is at the same time an unital algebra (H, μ_H, η_H) and a counital coalgebra $(H, \Delta_H, \epsilon_H)$ which are compatible by the Hopf compatibility law which includes the switch map σ . The multiplication μ_H extends to a multiplication $\mu_{H\otimes H}$.

$$(3-32) \Delta_H \mu_H = \mu_{H \otimes H} (\Delta_H \otimes \Delta_H) \mu_{H \otimes H} := (\mu_H \otimes \mu_H) (\operatorname{Id} \otimes \sigma \otimes \operatorname{Id})$$

Hence Δ_H is an algebra morphism $(H,\mu_H) \to (H \otimes H,\mu_{H\otimes H})$. A Hopf algebra unifies the concept of a (Lie-)group and a (Lie-)algebra at the same time. An element $g \in H$ is called group like if $\Delta(g) = g \otimes g$, that is the diagonal action or 'copying'. An element $p \in H$ is called primitive (or algebra like) if $\Delta(p) = p \otimes 1 + 1 \otimes p$. Primitive elements generate for example the (universal enveloping) Lie algebra of a Lie group seen as Hopf algebra. This analogy extends to the action of a Hopf algebra on a module M. Hence we have a H-action $H \otimes M \to M$ and a H-coaction $M \to H \otimes M$, which need to fulfill an analogue of the Hopf compatibility law. In graphical terms this reads, using white nodes for Hopf co/multiplications, as follows

The multiplication map $\mu_{H\otimes H}$ is depicted as the dashed box, modules receive bold lines. The graphical description makes it clear, that one can interchange the role of multiplication and co-multiplication and we see that μ_H is a morphism of coalgebras $(H\otimes H, \Delta_{H\otimes H})$ and (H, Δ_H)

(3-34)
$$\Delta_H \mu_H = (\mu_H \otimes \mu_H) \Delta_{H \otimes H} \qquad \Delta_{H \otimes H} := (\operatorname{Id} \otimes \sigma \otimes \operatorname{Id})(\Delta_H \otimes \Delta_H)$$

just moving the dashed box in the tangle up. The compatibility law for a Hopf action on a left H-co/module M has a bold rightmost line in the lhs of (3-33).

The conditions under which a finite Hopf algebra over a commutative ring R is Frobenius were worked out by (Larson & Sweedler 1969) for R a principal ideal domain, and by (Pareigis 1971) for general R with $\operatorname{Pic}[R] = 0$ (The Abelian Picard group $\operatorname{Pic}[R]$ consists of the set of isoclasses [X] of linebundles X over R (i.e. $X^* \otimes X \cong R$) with \otimes as multiplication $[X] + [X'] = [X \otimes X']$ and $X \mapsto X^* = \operatorname{Hom}(X, R)$ as inverse). These results have consequences for the existence and invertibility of the antipode, which in turn is relevant for Hopf algebra cohomology and invariants of 3-manifolds. So called integrals provide the main tool to prove these facts. Integrals allow one to construct Frobenius homomorphisms and equivalently the Frobenius bilinear form.

Let $\epsilon: H \to R$ be the augmentation map, which is an algebra homomorphism $\epsilon(ab) = \epsilon(a)\epsilon(b)$.

Definition 3.13: A right (left) integral is a $\mu_r \in H$ ($\mu_l \in H$) satisfying for all $h \in H$ the relation $h\mu_r = \epsilon(h)\mu_r$ ($\mu_l h = \epsilon(h)\mu_l$). The space of all right (left) integrals is denoted as

(3-35)
$$\int_{H}^{r} := \{ \mu_r \in H \mid \forall h \in H. \, h\mu_r = \epsilon(h)\mu_r \}$$
$$(\int_{H}^{l} := \{ \mu_l \in H \mid \forall h \in H. \, \mu_l = \epsilon(h)\mu_l \})$$

Graphically integrals look like

$$(3-36) \qquad \qquad \underbrace{\bullet}_{\epsilon} = \underbrace{\bullet}_{\epsilon} ; \qquad \underbrace{\mu^r}_{\mu^r} = \underbrace{\bullet}_{\mu^r} ; \qquad \underbrace{\mu^l}_{\mu^l} = \underbrace{\bullet}_{\mu^l}$$

Let H be a finitely generated projective module over a commutative ring with $\operatorname{Pic}[R]=0$ underlying a Hopf algebra H. (The mild condition $\operatorname{Pic}[R]=0$ can be lifted by studying quasi Frobenius rings, which we do not pursue, e.g. (Nicholson & Yousif 2003).) The crucial property we need to construct a Frobenius homomorphism is the fact hat \int_H^r is a one dimensional module over R

As $\int_H^r \cong {}_RR$ as an R-module is one dimensional, it is an invertible module. The following theorem, taken from (Lorenz 2011), summarizes the work of (Larson & Sweedler 1969, Pareigis 1971, Oberst & Schneider 1973).

Theorem 3.14: Let H be a finite projectively generated Hopf algebra over the commutative ring R, then:

- The antipode S is bijective (has a linear inverse). This implies $\int_H^r = S(\int_H^t)$.
- H is a Frobenius R-algebra iff $\int_H^r \cong R$. This holds true if Pic[R] = 0. Moreover, if H is Frobenius, then the dual Hopf algebra H^* is Frobenius too.
- \bullet Let H be Frobenius. Then H is symmetric iff

- (i) H is unimodular (i.e. $\int_H^r = \int_H^l$), and (ii) \mathbf{S}^2 is an inner automorphism of H.

The existence of a one-dimensional R-module of right/left integrals entails the construction of the Frobenius isomorphism $\beta^{\bullet}: H \to H^*$ using the Hopf algebra structure on H. The following tangle diagrams explain how to construct the Frobenius homomorphism, and Frobenius isomorphisms, out of an unimodular integral $\Lambda = \mu_l = \mu_r$ and the Hopf algebra structure:

$$(3-38) \qquad \begin{array}{c} H^*\!H^* \\ \overline{\beta} \end{array} := \begin{array}{c} H^*\!H^* \\ \overline{\beta} \end{array} := \begin{array}{c} H^*\!H^*\!H \\ \overline{\beta} \end{array} := \begin{array}{c} H^*\!H^*\!H \\ \overline{\beta} \end{array} := \begin{array}{c} G^* \\ \overline{\beta} \end{array} :=$$

The first equation in (3-38) defines the (dual) bilinear form from the integral Λ . The second equation defines right orthogonality of the form, see (Larson & Sweedler 1969), we need left orthogonality too. Orthogonality guarantees that the Frobenius isomorphism respects the module structure. The third equation defines the element u, which is needed to show the existence of the antipode in (3-39). Finally the right equation in (3-39) defines the left action of H on a dual module ${}_HM^*$ using the antipode and the right action on M_H^* , which by duality (3-1) comes from the left action ${}_{H}M$, establishing an Frobenius isomorphism.

3.10. Information flow: Frobenius versus closed structures. A difference we hit over and over again is how the 'yanking' is realized in the closed and Frobenius situations. If we interpret the orientation of tanlges as information flow, then a module propagates information in time (downwards), while a dual module propagates information backwards in time (upwards). A further difference is, that the closed structure is characterized by a universal property, while the Frobenius structure depends on a *choice* of a bilinear form.

If we look at teleportation protocols modelled by 'yanking', the ability of Alice to chose between different Bell measurements indicates that she is not dealing with a unique map, but with a Frobenius structure. Also the creation of a shared entangled state is not unique, as every Bell state does the trick. The need to communicate classical information to Bob emerges from the need to communicate this choice of one of the four Bell states in an a priory mutually agreed on classical basis, which can be described by a special Frobenius algebra. Hence the 'yanking' in teleportation should be thought of as a Frobenius algebra related property.

A similar situation emerges in canonical quantization of fields. The negative frequency parts are interpreted not as 'particles' (field modes) travelling backwards in time, but as 'antiparticles' propagating homochronos. This doubling of field modes resembles then the Frobenius type information flow as discussed for teleportation. The β -multiplication is then given via the reproducing kernel property of the field propagators $\psi(x) = \int_Y g(x,y)\psi(y)$, but for a continuum. This calls for the extension of Frobenius structures to a non-unital situation, as on an infinite dimensional space one cannot have a unit, see (Abramsky & Heunen 2010, Coecke & Heunen 2011).

With regard to the topic of this book, it is worth looking at the vector space semantics of meanings of natural languages. There one has a set of grammatical types of vector spaces. The meaning of words are represented by vectors in differently typed vector spaces for nouns, transitive verbs, etc. On top of these types one has a Lambeck pregroup, see (Lambek 1999, Preller & Sadrzadeh 2011, Sadrzadeh, Coecke & Clark 2010, Preller 2012) and elsewhere in this volume, which is weakening the closed structure we are using here. Using a^l, a^r for left and right adjoints (duals), and having an order on the 'tensor monoid', one ends up, among others, with relations of the type $a^l a \le 1 \le aa^l$ and $aa^r \le 1 \le a^r a$. The later papers in *loc. cit.* show the striking similarity between this structure and categorical quantum mechanics, especially teleportation. With respect to our comment above, we think it should be investigated how this pregroup approach relates to the Frobenius setup. First we note that any corpus of words is finite, hence we can assume a good duality or a Frobenius structure on its freely generated vector spaces of types. As the order of words in sentences is crucial for their meaning, one has to deal with non-symmetric Froebnius algebras, also to prevent left and right duality to coincide. Hence one needs to take the Nakayama automorphism into account. As we have seen in section 3.8 the braiding and the left/right Frobenius isomorphisms can be transformed into one another (3-31). The pictures in (Preller & Sadrzadeh 2011, page 150) (and preface of this book) describing sentences as John likes Mary or John does not like Mary would then be replaced by tree like structures, where the multiplications are Frobenius pairings on the types, like $V \otimes W \rightarrow J$ for a transitiv verb. The order structure may call for lax-Frobenius structures. We close this speculation with the remark, that in phylogenetic biomathematics one encounters very similar problems. Namely to reconstruct ancestral relationships from present date gene sequences. This is an analogue process to the linguistic setup, reconstructing a tree which has as 'words' the preset day gene expression and as 'semantic meaning' the ancestral relation including branching times, see for example (Jarvis, Bashford & Sumner 2005, Draisma & Kuttler 2009, Jarvis & Sumner 2011, Sumner, Holland & Jarvis 2011).

4. A FEW POINTERS TO FURTHER LITERATURE

Frobenius algebras emerge in a large number of situations which we had no opportunity to discuss here. We give a few hints where such developments are found, and what problems they are addressing. Our pick on the literature is subjective with an edge towards graphically minded work.

Starting with (Abramsky & Coecke 2004) compact closed (dagger) categories were used to analyze quantum theory, and especially quantum protocols. Graphical methods have been utilized very heavily and led to 'picturalism' in quantum theory (Coecke & Duncan 2011). The usage of *several* Frobenius algebra structures at the same time (red-green-/ black-white-/ rgb-calculi, which will be discussed elsewhere in this volume) have captured, among other things, the concept of orthogonal bases (Coecke et al. 2012, Coecke et al. 2008), complementary bases,

the dagger structure, and such things as connectedness or disconnectedness of product states (e.g. entanglement for Bell, GHZ and W states). Specialness of Frobenius algebras is not found in invariant rings, leading to non-trivial relations between reciprocal bases and dual bases, but still Frobenius structures can be employed there (Khovanov 1997). The product of group characters can be understood along these lines, and the reader is invited to stare at (Macdonald 1979, 2nd ed. pp305–309) to see how it can be written in terms of the graphical calculus used here. A ring extension $\mathbb{Q}[q,t]/\mathbb{Q}$ leads to Macdonald polynomials. Below we will, however, give a pointer showing that it is not straight forward to generalize this 'Cartesian' setup (also needed for canonical quantization) to manifolds and general coordinate invariance.

A theme which is amenable to graphical treatment is the relation between Frobenius structures of iterated ring extensions (or towers of algebras) and the Jones polynomial, see (Kadison 1999, Khovanov 1997, Müger 2003). If seen as ring extensions $A_n/A_{n-1}/\dots/A_0$ the various extensions provide separability idempotents, allowing the introduction of the relevant Markov traces and finally of the Jones polynomial.

A further theme is the relation of characters and certain trace modules with Frobenius structures. A character χ_V of a module V in $A\operatorname{-mod}_R$ is given by

(4-1)
$$\chi_V(a) = \operatorname{Tr}_{V/R}(a_V) \in R, \qquad (a \in A)$$

where $a_V \in \text{End}(V)$ is given by $a_V(v) = av$. Hence characters form a subset of trace forms

$$\chi_v \in \frac{A^*}{[A^*, A^*]} \subseteq A^*$$

which vanish on [A, A] (are constant on orbits). This allows one, for example, to define the Higman trace $\tau = \tau_{\beta}$ for a Frobenius algebra A as

(4-3)
$$\tau_{\beta}: A \to A :: a \mapsto \sum_{i} x_{i} a y_{i}$$

independent on the choice of generators. τ is $\mathcal{Z}(A)$ linear, with $\mathcal{Z}(A)$ the center of A. For a matrix algebra $A=M_n(R)$ one finds $\tau(a)=\operatorname{trace}(a)1_{n\times n}$, and for a group algebra $A:=\mathbb{C}G$ one obtains the averaging or (up to normalization) Reynolds operator $\tau(a)=\sum_{g\in G}gag^{-1}$.

Furthermore, one can define the Casimir operator, equivalent to the Higman trace if A is symmetric, as

$$(4-4) c: A \to \mathcal{Z}(A) :: a \mapsto \sum_{i} y_{i} a x_{i}$$

which has deep connections to the Grothendieck groups, K-theory, and restriction and induction functors, for example see (Lorenz & Fitzgerald Tokoly 2010, Lorenz 2011) and references given there. It is remarkable that the related diagrams, easily drawn, of the Higman trace, or the Casimir operator emerge rather naturally in graphical calculations with Frobenius algebras.

As we mentioned already topological quantum field theory, we just remark that Frobenius structures play a prominent role in TQFT, see (Atiyah 1989, Kock 2003). The general idea to

apply a functor with codomain a tensor monoid of vector spaces has proved to be very versatile. Similar constructions can be found in the theory of vertex operators and rational conformal quantum field theories, see for example (Fuchs, Runkel & Schweigert 2002, Fuchs, Runkel & Schweigert 2007, Barmeier, Fuchs, Runkel & Schweigert 2010), which make heavy use of graphical calculi and provide further references.

A theme related to classical physics, is that of Frobenius manifolds, see (Hitchin 1997). Suppose M is a manifold of dimension n, one can impose the existence of the following sections

$$(4-5) \theta \in C^{\infty}(T^*M) g \in C^{\infty}(S^2T^*M) c \in C^{\infty}(S^3T^*M)$$

on the (co)tangent space of M. Here g is a metric on M, with covariant derivative ∇ , and θ is a 1-form (related to the the Frobenius homomorphism) and c is a symmetric rank 3 tensor. Let $\{e_i\}$ be an orthogonal basis of TM_x , with respect to the scalar product $(e_i,e_i)=\pm 1$, diagonalizing the left regular representation \mathbf{l}_a for $a\in TM_x$. After rescaling, the e_i are mutually annihilating idempotents. Let $e=\sum_i e_i$ be the unit of this algebra, then we get $\theta(v)=(e,v)$ and (u,v)=c(e,u,v). Let $\mu_i=\theta(e_i)$ and let $f_i\in TM_x^*$ be the dual basis of the e_i , then the structure maps can be written as

(4-6)
$$\theta = \sum \mu_i f_i \qquad \qquad g = \sum \mu_i f_i f_i \qquad \qquad c = \sum \mu_i f_i f_i f_i$$

The question which arises is, if this structure is compatible with the differential structure, that is the covariant derivative, defined by the metric. This compatibility leads to Chazy's *non-linear* differential equation, see *loc. cit.*. Chazy's equation provides a 'potential' for θ , g and c resulting in the proper definition of Frobenius manifolds.

If the metric is an Egoroff metric, having a sort of potential form, $g = \sum_i \mu_i f_i f_i = \sum_i \frac{\partial \phi}{\partial x_i} \mathrm{d} x_i^2$, then ∇c is symmetric, which implies $\mathrm{d}_A c = 0$. This result relates orthogonal coordinates (bases) to Frobenius structures, as we have seen in the quantum information setting. However, on a manifold we need to be careful, as not all orthogonal frames are allowed. This can be seen by the following example: The metric $\mathrm{d} x^2 + \mathrm{d} y^2$ defines a Frobenius structure on \mathbb{R}^2 , satisfying the condition above with $\phi = x_1 + x_2$. The same (in $\mathbb{R}^2 \setminus \{0\}$) metric in polar coordinates $\mathrm{d} r^2 + r^2 \mathrm{d} \phi^2$ does not carry a Frobenius structure with ∇c symmetric. This example shows clearly, that general coordinate transformations, hence general covariance, are not compatible with (symmetric) Frobenius structures. It sheds also some light on the usage of special Frobenius algebras in the semantics of quantum protocols as mentioned above.

Frobenius algebras can help to construct solutions of the Yang-Baxter equation (Beidar, Fong & Stolin 1997) (which is related to our remark about the Jones polynomial, and Kadison's work). In a boarder sense, one wants to study entwined modules, that is, modules $A \otimes C$ where A has an algebra structure and C has a coalgebra structure. It turns out, that Frobenius functors play a crucial role in studying such entwined modules. This is developed at length in (Caenepeel et al. 2002). It turns out, that this powerful algebraic tools allows one to attack non-linear differential equations and provide also solutions to the Yang-Baxter equation. A structure prominently used in solid state and high energy physics problems often termed there 'integrable models'.

We have in this work always assumed that the Frobenius algebra structure is associative. This can be relaxed to lax-Frobenius algebras or even general non-associative Frobenius algebras.

To my knowledge not much research has been done in this direction, however see (Oziewicz & Wene 2011) where a first attempt is made to study non-associative Frobenius algebras, also using graphical methods. Even in the seemingly trivial case of complex numbers, which can be seen as a Frobenius algebra over the reals (Kock 2003), one encounters different Frobenius algebras if associativity is dropped.

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